

# GROTHENDIECK QUASITOPOSES

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**ABSTRACT.** A full reflective subcategory  $\mathcal{E}$  of a presheaf category  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  is the category of sheaves for a topology  $j$  on  $\mathcal{C}$  if and only if the reflection from  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  into  $\mathcal{E}$  preserves finite limits. Such an  $\mathcal{E}$  is then called a Grothendieck topos. More generally, one can consider two topologies,  $j \subseteq k$ , and the category of sheaves for  $j$  which are also separated for  $k$ . The categories  $\mathcal{E}$  of this form for some  $\mathcal{C}$ ,  $j$ , and  $k$  are the Grothendieck quasitoposes of the title, previously studied by Borceux and Pedicchio, and include many examples of categories of spaces. They also include the category of concrete sheaves for a concrete site. We show that a full reflective subcategory  $\mathcal{E}$  of  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  arises in this way for some  $j$  and  $k$  if and only if the reflection preserves monomorphisms as well as pullbacks over elements of  $\mathcal{E}$ . More generally, for any quasitopos  $\mathcal{S}$ , we define a subquasitopos of  $\mathcal{S}$  to be a full reflective subcategory of  $\mathcal{S}$  for which the reflection preserves monomorphisms as well as pullbacks over objects in the subcategory, and we characterize such subquasitoposes in terms of universal closure operators.

## 1. INTRODUCTION

A Grothendieck topos is a category of the form  $\text{Sh}(\mathcal{C}, j)$  for a small category  $\mathcal{C}$  and a (Grothendieck) topology  $j$  on  $\mathcal{C}$ . These categories have been of fundamental importance in geometry, logic, and other areas. Such categories were characterized by Giraud as the cocomplete categories with a generator, satisfying various exactness conditions expressing compatibility between limits and colimits.

The category  $\text{Sh}(\mathcal{C}, j)$  is a full subcategory of the presheaf category  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ , and the inclusion has a finite-limit-preserving left adjoint. This in fact leads to another characterization of Grothendieck toposes. A full subcategory is said to be *reflective* if the inclusion has a left adjoint, and is said to be a *localization* if moreover this left adjoint preserves finite limits. A category is a Grothendieck topos if and only if it is a localization of some presheaf category  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  on a small category  $\mathcal{C}$ . (We shall henceforth only consider presheaves on small categories.)

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Elementary toposes, introduced by Lawvere and Tierney, generalize Grothendieck toposes; the non-elementary conditions of cocompleteness and a generator in the Giraud characterization are replaced by the requirement that certain functors, which the Giraud conditions guarantee are continuous, must in fact be representable. Yet another characterization of the Grothendieck toposes is as the elementary toposes which are locally presentable, in the sense of [10]. We cite the encyclopaedic [12, 13] as a general reference for topos-theoretic matters.

A *quasitopos* [14] is a generalization of the notion of elementary topos. The main difference is that a quasitopos need not be balanced: this means that in a quasitopos a morphism may be both an epimorphism and a monomorphism without being invertible. Rather than a classifier for all subobjects, there is only a classifier for *strong* subobjects (see Section 2 below). The definition, then, of a quasitopos is a category  $\mathcal{E}$  with finite limits and colimits, for which  $\mathcal{E}$  and each slice category  $\mathcal{E}/E$  of  $\mathcal{E}$  is cartesian closed, and which has a classifier for strong subobjects. A simple example of a quasitopos which is not a topos is a Heyting algebra, seen as a category by taking the objects to be the elements of the Heyting algebra, with a unique arrow from  $x$  to  $y$  just when  $x \leq y$ . Other examples include the category of convergence spaces in the sense of Choquet, or various categories of differentiable spaces, studied by Chen. See [12] once again for generalities about quasitoposes, and [1] for the examples involving differentiable spaces.

The notion of Grothendieck quasitopos was introduced in [2]. Once again, there are various possible characterizations:

- (i) the locally presentable quasitoposes;
- (ii) the locally presentable categories which are locally cartesian closed and in which every strong equivalence relation is the kernel pair of its coequalizer;
- (iii) the categories of the form  $\text{Sep}(k) \cap \text{Sh}(j)$  for topologies  $j$  and  $k$  on a small category  $\mathcal{C}$ , with  $j \subseteq k$ .

In (ii), an equivalence relation in a category  $\mathcal{E}$  is a pair  $d, c: R \rightrightarrows A$  inducing an equivalence relation  $\mathcal{E}(X, R)$  on each hom-set  $\mathcal{E}(X, A)$ ; it is said to be *strong* if the induced map  $R \rightarrow A \times A$  is a strong monomorphism. In (iii), we write  $\text{Sh}(j)$  for the sheaves for  $j$ , and  $\text{Sep}(k)$  for the category of separated objects for  $k$ ; these are defined like sheaves, except that in the sheaf condition we ask only for the uniqueness, not the existence, of the gluing. A category  $\mathcal{C}$  equipped with topologies  $j$  and  $k$  with  $j \subseteq k$  is called a *bisite* in [13], and a presheaf on  $\mathcal{C}$  which is a  $j$ -sheaf and  $k$ -separated is then said to be  $(j, k)$ -*biseparated*.

A special case is where  $\mathcal{C}$  has a terminal object and the representable functor  $\mathcal{C}(1, -)$  is faithful, and  $k$  is the topology generated by the covering families consisting, for each  $C \in \mathcal{C}$ , of the totality of maps  $1 \rightarrow C$ . If  $j$  is any subcanonical topology contained in  $k$ , then  $(\mathcal{C}, j)$  is a concrete site in the sense of [1] (see also [8, 9]) for which the concrete sheaves are exactly the  $(j, k)$ -biseparated presheaves.

In the case of Grothendieck toposes, a full reflective subcategory of a presheaf category  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  has the form  $\text{Sh}(j)$  for some (necessarily unique) topology  $j$  if and only if the reflection preserves finite limits. The lack of a corresponding result for Grothendieck quasitoposes is a noticeable gap in the existing theory, and it is precisely this gap which we aim to fill.

It is well-known that the reflection from  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  to  $\text{Sep}(k) \cap \text{Sh}(j)$  preserves finite products and monomorphisms. In Example 3.9 below, we show that this does not suffice to characterize such reflections, using the reflection of directed graphs into preorders as a counterexample. We provide a remedy for this in Theorem 6.1, where we show that a reflection  $L: [\mathcal{C}^{\text{op}}, \mathbf{Set}] \rightarrow \mathcal{E}$  has this form for topologies  $j$  and  $k$  if and only if  $L$  preserves finite products and monomorphisms and is also *semi-left-exact*, in the sense of [6]. Alternatively, such  $L$  can be characterized as those which preserve monomorphisms and have *stable units*, again in the sense of [6]. The stable unit condition can most easily be stated by saying that  $L$  preserves all pullbacks in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  over objects in the subcategory  $\mathcal{E}$ . For each object  $X \in \mathcal{E}$ , the slice category  $\mathcal{E}/X$  is a full subcategory of  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]/X$ , with a reflection  $L_X: [\mathcal{C}^{\text{op}}, \mathbf{Set}]/X \rightarrow \mathcal{E}/X$  given on objects by the action of  $L$  on a morphism into  $X$ ; the condition that  $L$  preserve all pullbacks over objects of  $\mathcal{E}$  is equivalently the condition that each  $L_X$  preserve finite products.

Since a *subtopos* of a topos  $\mathcal{S}$  is by definition a full reflective subcategory of  $\mathcal{S}$  for which the reflection preserves finite limits, the Grothendieck toposes are precisely the subtoposes of presheaf toposes. Subtoposes of an arbitrary topos can be characterized in terms of Lawvere-Tierney topologies; more importantly for our purposes, they can be characterized in terms of universal closure operators.

By analogy with this case, we define a *subquasitopos* of a quasitopos  $\mathcal{S}$  to be a full reflective subcategory of  $\mathcal{S}$  for which the reflection preserves monomorphisms and has stable units. Thus a Grothendieck quasitopos is precisely a subquasitopos of a presheaf topos. We also give a characterization of subquasitoposes of an arbitrary quasitopos  $\mathcal{S}$ , using universal closure operators.

We begin, in the following section, by recalling a few basic notions that will be used in the rest of the paper; then in Section 3 we study various weakenings of finite-limit-preservation for a reflection, and the relationships between these. In Section 4 we study conditions under which reflective subcategories of quasitoposes are quasitoposes. In Section 5 we characterize subquasitoposes of a general quasitopos, before turning, in Section 6, to subquasitoposes of presheaf toposes and their relationship with Grothendieck quasitoposes.

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## 2. PRELIMINARIES

We recall a few basic notions that will be used in the rest of the paper.

A monomorphism  $m: X \rightarrow Y$  is said to be *strong* if for all commutative diagrams

$$\begin{array}{ccc} X' & \xrightarrow{e} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{m} & Y \end{array}$$

with  $e$  an epimorphism, there is a unique map  $Y' \rightarrow X$  making the two triangles commute. *Strong epimorphisms* are defined dually. A strong epimorphism which is also a monomorphism is invertible, and dually a strong monomorphism which is also an epimorphism is invertible.

A *weak subobject classifier* is a morphism  $t: 1 \rightarrow \Omega$  with the property that for any strong monomorphism  $m: X \rightarrow Y$  there is a unique map  $f: Y \rightarrow \Omega$  for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \downarrow & & \downarrow f \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

is a pullback.

A category with finite limits is said to be *regular* if every morphism factorizes as a strong epimorphism followed by a monomorphism, and

if moreover any pullback of a strong epimorphism is again a strong epimorphism. It then follows that the strong epimorphisms are precisely the *regular epimorphisms*; that is, the morphisms which are the coequalizer of some pair of maps. Our regular categories will always be assumed to have finite limits.

A full subcategory is *reflective* when the inclusion has a left adjoint; this left adjoint is called the *reflection*.

Throughout the paper,  $\mathcal{S}$  will be a category with finite limits; later on we shall make further assumptions on  $\mathcal{S}$ , such as being regular; when we finally come to our characterization of Grothendieck quasitoposes,  $\mathcal{S}$  will be a presheaf topos.

Likewise, throughout the paper,  $\mathcal{E}$  will be a full reflective subcategory of  $\mathcal{S}$ . We shall write  $L$  for the reflection  $\mathcal{S} \rightarrow \mathcal{E}$  and also sometimes for the induced endofunctor of  $\mathcal{S}$ , and we write  $\ell: 1 \rightarrow L$  for the unit of the reflection. It is convenient to assume that the inclusion  $\mathcal{E} \rightarrow \mathcal{S}$  is replete, in the sense that any object isomorphic to one in the image is itself in the image. It is also convenient to assume that  $\ell A: A \rightarrow LA$  is the identity whenever  $A \in \mathcal{E}$ . Neither assumption affects the results of the paper.

We shall say that the reflection has *monomorphic units* if each component  $\ell X: X \rightarrow LX$  of the unit is a monomorphism, with an analogous meaning for *strongly epimorphic units*. When  $L$  preserves finite limits it is said to be a *localization*.

An object  $A$  of a category  $\mathcal{C}$  is said to be *orthogonal* to a morphism  $f: X \rightarrow Y$  if each  $a: X \rightarrow A$  factorizes uniquely through  $f$ . If instead each  $a: X \rightarrow A$  factorizes in at most one way through  $f$ , the object  $A$  is said to be *separated with respect to  $f$* , or  *$f$ -separated*. If  $\mathcal{F}$  is a class of morphisms, we say that  $A$  is  $\mathcal{F}$ -orthogonal or  $\mathcal{F}$ -separated if it is  $f$ -orthogonal or  $f$ -separated for each  $f \in \mathcal{F}$ .

### 3. LIMIT-PRESERVING CONDITIONS FOR REFLECTIONS

In this section we study various conditions on a reflection  $L: \mathcal{S} \rightarrow \mathcal{E}$  weaker than being a localization. First observe that any reflective subcategory is closed under limits, so the terminal object of  $\mathcal{S}$  lies in  $\mathcal{E}$ , and so  $L$  always preserves the terminal object. Thus preservation of finite limits is equivalent to preservation of pullbacks; our conditions all say that *certain* pullbacks are preserved.

*Preservation of finite products.* Since  $L$  preserves the terminal object, preservation of finite products amounts to preservation of binary products, or to preservation of pullbacks over the terminal object.

By a well-known result due to Brian Day [7], if  $\mathcal{S}$  is cartesian closed, then  $L$  preserves finite products if and only if  $\mathcal{E}$  is an exponential ideal in  $\mathcal{S}$ ; it then follows in particular that  $\mathcal{E}$  is cartesian closed.

*Stable units.* For each object  $B \in \mathcal{E}$ , the reflection  $L: \mathcal{S} \rightarrow \mathcal{E}$  induces a reflection  $L_B: \mathcal{S}/B \rightarrow \mathcal{E}/B$  onto the full subcategory  $\mathcal{E}/B$  of  $\mathcal{S}/B$ . The original reflection  $L$  is said to have *stable units* when each  $L_B$  preserves finite products, or equivalently when  $L$  preserves all pullbacks over objects of  $\mathcal{E}$ . Since the terminal object lies in  $\mathcal{E}$ , this implies in particular that  $L$  preserves finite products.

If  $\mathcal{S}$  is locally cartesian closed then, by the Day reflection theorem [7] again,  $L$  has stable units just when each  $\mathcal{E}/B$  is an exponential ideal in  $\mathcal{S}/B$ ; it then follows that  $\mathcal{E}$  is locally cartesian closed.

The name stable units was originally introduced in [6] for an apparently weaker condition, namely that  $L$  preserve each pullback of the form

$$\begin{array}{ccc} P & \xrightarrow{q} & A \\ p \downarrow & & \downarrow u \\ X & \xrightarrow{\ell_X} & LX \end{array}$$

but it was observed in [4, Section 3.7] that these two conditions are in fact equivalent. Notice also that since  $L\ell_X$  is invertible, to say that  $L$  preserves the pullback is equivalently to say that  $L$  inverts  $q$ .

*Frobenius.* We say that  $L$  satisfies the *Frobenius condition* when it preserves products of the form  $X \times A$ , with  $A \in \mathcal{E}$ .

The condition is often given in the more general context of an adjunction, not necessarily a reflection, between categories with finite products. In this case, the condition is that the canonical map

$$\varphi: L(X \times A) \rightarrow LX \times A,$$

defined using the comparison  $L(X \times IA) \rightarrow LX \times LIA$  and the counit  $LIA \rightarrow A$ , should be invertible.

As is well-known, if  $\mathcal{E}$  and  $\mathcal{S}$  are both cartesian closed, then this condition is equivalent to the right adjoint  $I: \mathcal{E} \rightarrow \mathcal{S}$  preserving internal homs. As is perhaps less well-known, in our setting of a reflection it is enough to assume that  $\mathcal{S}$  is cartesian closed, and then the condition ensures that the internal homs restrict to  $\mathcal{E}$ : see Proposition 4.2 below. Thus if  $\mathcal{S}$  is cartesian closed, then  $L$  satisfies the Frobenius condition if and only if  $\mathcal{E}$  is closed in  $\mathcal{S}$  under internal homs.

In fact, for any monadic adjunction satisfying the Frobenius condition, internal homs may be lifted along the right adjoint. More generally still, there is a version of the Frobenius condition defined for monoidal categories in which the tensor product is not required to be the product, and once again the internal homs can be lifted along the right adjoint: see [3, Proposition 3.5 and Theorem 3.6].

*Semi-left-exact.* We say, following [6], that  $L$  is *semi-left-exact* if it preserves each pullback

$$\begin{array}{ccc} P & \xrightarrow{q} & A \\ p \downarrow & & \downarrow u \\ X & \xrightarrow{\ell_X} & LX \end{array}$$

with  $A \in \mathcal{E}$ . This is clearly implied by the stable units condition. By [6, Theorem 4.3] it in fact implies, and so is equivalent to, the apparently stronger condition that  $L$  preserve each pullback of the form

$$\begin{array}{ccc} P & \xrightarrow{q} & A \\ p \downarrow & & \downarrow u \\ X & \xrightarrow{v} & B \end{array}$$

with  $A$  and  $B$  in  $\mathcal{E}$ . But this latter condition is in turn equivalent to the condition that each  $L_B: \mathcal{S}/B \rightarrow \mathcal{E}/B$  be Frobenius. Thus we see that semi-left-exactness is in fact a “localized” version of the Frobenius condition. In particular, we may take  $B = 1$ , and see that semi-left-exactness implies the Frobenius condition.

Once again, if  $\mathcal{S}$  is locally cartesian closed, so that each  $\mathcal{S}/B$  is cartesian closed, then  $L$  is semi-left-exact just when each  $\mathcal{E}/B$  is closed in  $\mathcal{S}/B$  under internal homs. This implies that each  $\mathcal{E}/B$  is cartesian closed, and so that  $\mathcal{E}$  is locally cartesian closed; see Lemma 4.3 below.

*Preservation of monomorphisms.* Preservation of monomorphisms will also be an important condition in what follows. Once again it can be seen as preservation of certain pullbacks. Notice also that  $L: \mathcal{S} \rightarrow \mathcal{E}$  satisfies this condition if and only if each  $L_B: \mathcal{S}/B \rightarrow \mathcal{E}/B$  does so.

*Relationships between the conditions.* We summarize in the diagram

$$\begin{array}{ccc} \text{stable units} & \Longrightarrow & \text{semi-left-exact} & \text{mono-preserving} \\ \Downarrow & & \Downarrow & \Downarrow \\ \text{finite-product-preserving} & \Longrightarrow & \text{Frobenius} & \text{mono-preserving} \end{array}$$



the relationships found so far between these conditions. Each condition in the top row amounts to requiring the condition below it to hold for all  $L_B : \mathcal{S}/B \rightarrow \mathcal{E}/B$  with  $B \in \mathcal{E}$ .

In Theorem 3.5 below, we shall see that if  $\mathcal{S}$  is regular and  $L$  preserves monomorphisms, then the stable units condition is equivalent to the conjunction of the semi-left-exactness and finite-product-preserving conditions. In order to prove this, we start by considering separately the case where the components of the unit  $\ell : 1 \rightarrow L$  are monomorphisms and that where they are strong epimorphisms.

**Theorem 3.1.** *If  $\mathcal{S}$  is finitely complete and the reflection  $L : \mathcal{S} \rightarrow \mathcal{E}$  has monomorphic units, then the following are equivalent:*

- (i)  *$L$  preserves finite limits;*
- (ii)  *$L$  has stable units;*
- (iii)  *$L$  is semi-left-exact and preserves finite products.*

*Proof.* The implication (i)  $\Rightarrow$  (ii) is trivial, while (ii)  $\Rightarrow$  (iii) was observed above. Thus it remains to verify the implication (iii)  $\Rightarrow$  (i).

Suppose then that  $L$  is semi-left-exact and preserves finite products. We must show that it preserves equalizers. Given  $f, g : Y \rightrightarrows Z$  in  $\mathcal{S}$ , form the equalizer  $e : X \rightarrow Y$  of  $f$  and  $g$ , and the equalizer  $d : A \rightarrow LY$  of  $Lf$  and  $Lg$ ; of course  $A \in \mathcal{E}$  since  $\mathcal{E}$  is closed in  $\mathcal{S}$  under limits. There is a unique map  $k : X \rightarrow A$  making the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{e} & Y & \xrightarrow{f} & Z \\
 \downarrow k & & \downarrow \ell Y & \xrightarrow{Lg} & \downarrow \ell Z \\
 A & \xrightarrow{d} & LY & \xrightarrow{Lf} & LZ
 \end{array}$$

commute. It follows easily from the fact that  $\ell Z$  is a monomorphism that the square on the left is a pullback. Since  $A \in \mathcal{E}$ , it follows by semi-left-exactness that  $L$  inverts  $k$ , which is equivalently to say that  $L$  preserves the equalizer of  $f$  and  $g$ .  $\square$

**Theorem 3.2.** *If  $\mathcal{S}$  is regular, and the reflection  $L : \mathcal{S} \rightarrow \mathcal{E}$  has strongly epimorphic units, then the following are equivalent:*

- (i)  *$L$  preserves finite products and monomorphisms;*
- (ii)  *$L$  has stable units and preserves monomorphisms.*

*Proof.* Since having stable units always implies the preservation of finite products, it suffices to show that (i) implies (ii). Suppose then that  $L$



preserves finite products and monomorphisms. Let

$$\begin{array}{ccc} P & \xrightarrow{q} & Y \\ p \downarrow & & \downarrow u \\ X & \xrightarrow{\ell X} & LX \end{array}$$

be a pullback. Since  $\ell X$  is a strong epimorphism, so is its pullback  $q$ ; but the left adjoint  $L$  preserves strong epimorphisms and so  $Lq$  is also a strong epimorphism in  $\mathcal{E}$ .

Since  $L$  preserves finite products and monomorphisms, it preserves jointly monomorphic pairs; thus  $Lp$  and  $Lq$  are jointly monomorphic. It follows that the canonical comparison from  $LP$  to the pullback of  $Lu$  and  $L\ell X$  is a monomorphism, but this comparison is just  $Lq$ . Thus  $Lq$  is a strong epimorphism and a monomorphism, and so invertible. This proves that  $L$  has stable units.  $\square$

Having understood separately the case of a reflection with monomorphic units and that of one with strongly epimorphic units, we now combine these to deal with the general situation. The first step in this direction is well-known; see [6] for example.

**Proposition 3.3.** *Suppose that  $\mathcal{S}$  is regular. If  $\mathcal{R}$  is the closure of  $\mathcal{E}$  in  $\mathcal{S}$  under subobjects, then the reflection of  $\mathcal{S}$  into  $\mathcal{E}$  factorizes as*

$$\mathcal{E} \begin{array}{c} \xleftarrow{\overline{L}} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathcal{R} \begin{array}{c} \xleftarrow{L'} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathcal{S}$$

where  $L'$  has strongly epimorphic units and  $\overline{L}$  has monomorphic units.

*Proof.* A straightforward argument shows that an object  $X \in \mathcal{S}$  lies in  $\mathcal{R}$  if and only if the unit  $\ell X: X \rightarrow LX$  is a monomorphism. Then the restriction  $\overline{L}: \mathcal{R} \rightarrow \mathcal{E}$  of  $L$  is clearly a reflection of  $\mathcal{R}$  into  $\mathcal{E}$ .

Since  $\mathcal{S}$  is regular we may factorize  $\ell: 1 \rightarrow L$  as a strong epimorphism  $\ell': 1 \rightarrow L'$  followed by a monomorphism  $\kappa: L' \rightarrow L$ . Since  $L'X$  is a subobject of  $LX$ , it lies in  $\mathcal{R}$ . We claim that  $\ell'X: X \rightarrow L'X$  is a reflection of  $X$  into  $\mathcal{R}$ . Given an object  $Y \in \mathcal{R}$ , the unit  $\ell Y: Y \rightarrow LY$  is a monomorphism, and now if  $f: X \rightarrow Y$  is any morphism, then in

the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\ell'X} & L'X \\
 \downarrow f & & \downarrow \kappa X \\
 & & LX \\
 & & \downarrow Lf \\
 Y & \xrightarrow{\ell Y} & LY
 \end{array}$$

$\ell'X$  is a strong epimorphism and  $\ell Y$  a monomorphism, so there is a unique induced  $g: L'X \rightarrow Y$  with  $g.\ell'X = f$  and  $\ell Y.g = Lf.\kappa X$ . This gives the existence of a factorization of  $f$  through  $\ell'X$ ; the uniqueness is automatic since  $\ell'X$  is a (strong) epimorphism.  $\square$

**Corollary 3.4.** *Suppose that  $\mathcal{S}$  is regular. If  $L: \mathcal{S} \rightarrow \mathcal{E}$  is semi-left-exact and preserves finite products and monomorphisms, then its restriction  $\bar{L}: \mathcal{R} \rightarrow \mathcal{E}$  to  $\mathcal{R}$  preserves finite limits, while  $L': \mathcal{S} \rightarrow \mathcal{R}$  has stable units and preserves monomorphisms.*

*Proof.* Since  $L$  is semi-left-exact and preserves finite products and monomorphisms, the same is true of its restriction  $\bar{L}$ . Thus  $\bar{L}$  preserves finite limits by Theorem 3.1 and the fact that  $\bar{L}$  has monomorphic units.

As for  $L'$ , since it has strongly epimorphic units it will suffice, by Theorem 3.2, to show that it preserves finite products and monomorphisms.

First observe that if  $m: X \rightarrow Y$  is a monomorphism in  $\mathcal{S}$ , then we have a commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{\ell'X} & L'X & \xrightarrow{\kappa X} & LX \\
 m \downarrow & & L'm \downarrow & & \downarrow Lm \\
 Y & \xrightarrow{\ell'Y} & L'Y & \xrightarrow{\kappa Y} & LY
 \end{array}$$

in which  $Lm$  and  $\kappa X$  are monomorphisms, and thus also  $L'm$ . This proves that  $L'$  preserves monomorphisms.

On the other hand, for any objects  $X, Y \in \mathcal{S}$ , we have a commutative diagram

$$\begin{array}{ccccc}
 X \times Y & \xrightarrow{\ell'(X \times Y)} & L'(X \times Y) & \xrightarrow{\kappa(X \times Y)} & L(X \times Y) \\
 \searrow \ell'X \times \ell'Y & & \downarrow \pi' & & \downarrow \pi \\
 & & L'X \times L'Y & \xrightarrow{\kappa X \times \kappa Y} & LX \times LY
 \end{array}$$

in which  $\pi$  and  $\pi'$  are the canonical comparison maps. Now  $\ell'X \times \ell'Y$  is a strong epimorphism, since in a regular category these are closed under products, and  $\kappa X \times \kappa Y$  is a monomorphism, since in any category these are closed under products. Since  $L$  preserves products,  $\pi$  is invertible, and it now follows that  $\pi'$  is also invertible. Thus  $L'$  preserves finite products.  $\square$

**Theorem 3.5.** *Let  $\mathcal{S}$  be regular, and  $L: \mathcal{S} \rightarrow \mathcal{E}$  an arbitrary reflection onto a full subcategory, with unit  $\ell: 1 \rightarrow L$ . Then the following are equivalent:*

- (i)  *$L$  is semi-left-exact, and preserves finite products and monomorphisms;*
- (ii)  *$L$  has stable units and preserves monomorphisms.*

*Proof.* The non-trivial part is that (i) implies (ii). Suppose then that  $L$  satisfies (i), and factorize  $L$  as  $\overline{L}L'$  as in Proposition 3.3. By Corollary 3.4, we know that  $\overline{L}$  preserves finite limits, while  $L'$  preserves pullbacks over objects of  $\mathcal{R}$  and monomorphisms, thus the composite  $\overline{L}L'$  preserves pullbacks over objects of  $\mathcal{R}$  and monomorphisms, and so in particular has stable units and preserves monomorphisms.  $\square$

**Remark 3.6.** In fact we have shown that a reflection  $L$  satisfying the equivalent conditions in the theorem preserves all pullbacks over an object of  $\mathcal{R}$ ; that is, over a subobject of an object in the subcategory  $\mathcal{E}$ .

Having described the positive relationships between our various conditions, we now show the extent to which they are independent. We shall give three examples; in each case  $\mathcal{S}$  is a presheaf category.

**Example 3.7.** Let  $\mathbf{2}$  be the full subcategory of  $\mathbf{Set} \times \mathbf{Set}$  consisting of the objects  $(0, 0)$  and  $(1, 1)$ . This is reflective, with the reflection sending a pair  $(X, Y)$  to  $(0, 0)$  if  $X$  and  $Y$  are both empty, and  $(1, 1)$  otherwise. It's easy to see that this is semi-left-exact and preserves monomorphisms, but fails to preserve the product  $(0, 1) \times (1, 0) = (0, 0)$  since  $L(0, 1) = L(0, 1) = 1$  but  $L(0, 0) = 0$ .

**Example 3.8.** Consider  $\mathbf{Set}$  as the full reflective subcategory of  $\mathbf{RGph}$  consisting of the discrete reflexive graphs. This time the reflector sends a graph  $G$  to its set of connected components  $\pi_0 G$ . This is well-known to preserve finite products. Furthermore, it preserves pullbacks over a discrete reflexive graph  $X$ , since  $\mathbf{Set}/X \simeq \mathbf{Set}^X$  and  $\mathbf{RGph}/X \simeq \mathbf{RGph}^X$  and the induced  $\pi_0/X: \mathbf{RGph}/X \rightarrow \mathbf{Set}/X$  is just  $\pi_0^X: \mathbf{RGph}^X \rightarrow \mathbf{Set}^X$ , which preserves finite products since  $\pi_0$  does so. Thus  $\pi_0$  has stable units, and so is semi-left-exact (and, as we have already seen, preserves finite products). But  $\pi_0$  does not preserve

monomorphisms: any set  $X$  gives rise both to a discrete reflexive graph and the “complete” reflexive graph  $KX$  with exactly one directed edge between each pair of vertices. The inclusion  $X \rightarrow KX$  is a monomorphism, but  $\pi_0 X$  is just  $X$ , while  $\pi_0 KX = 1$ . Thus  $\pi_0$  does not preserve this monomorphism if  $X$  has more than one vertex.

**Example 3.9.** Let **RGph** be the category of reflexive graphs, and **Preord** the full reflective subcategory of preorders. Since the reflection sends a graph  $G$  to a preorder on the set of vertices of  $G$ , it clearly preserves monomorphisms. An easy calculation shows that for any reflexive graph  $G$  and preorder  $X$ , the internal hom  $[G, X]$  in **RGph** again lies in **Preord**, corresponding to the set of graph homomorphisms equipped with the pointwise preordering. Thus **Preord** is an exponential ideal in **RGph**, and so the reflection preserves finite products by [7]. On the other hand, by Lemma 4.3 below, the reflection cannot be semi-left-exact since **Preord** is not locally cartesian closed. To see this, consider the preorder  $X = \{x, y, y', z\}$  with  $x \leq y$  and  $y' \leq z$ , and the two maps  $1 \rightarrow X$  picking out  $y$  and  $y'$ . Their coequalizer is  $\{x \leq y \leq z\}$ , but this is not preserved by pulling back along the inclusion of  $\{x \leq z\}$  into  $X$ , so **Preord** cannot be locally cartesian closed.

Our characterization of Grothendieck quasitoposes, in Theorem 6.1 below, involves three conditions on a reflection: that it be semi-left-exact, that it preserve finite products, and that it preserve monomorphisms. By the three examples above, we see that none of these three conditions can be omitted.

#### 4. QUASITOPPOSES

In this section we take a slight detour to study conditions under which a reflective subcategory is a quasitopos. First of all, a reflective subcategory  $\mathcal{E}$  of  $\mathcal{S}$  has any limits or colimits which  $\mathcal{S}$  does, so of course we have:

**Proposition 4.1.** *If  $L: \mathcal{S} \rightarrow \mathcal{E}$  is any reflection, then  $\mathcal{E}$  has finite limits and finite colimits if  $\mathcal{S}$  does so.*

To deal with the remaining parts of the quasitopos structure we require some assumptions on the reflection.

**Proposition 4.2.** *If  $L: \mathcal{S} \rightarrow \mathcal{E}$  is Frobenius then  $\mathcal{E}$  is cartesian closed if  $\mathcal{S}$  is so.*

*Proof.* Suppose that  $L$  satisfies the Frobenius condition. We shall show that if  $A, B \in \mathcal{E}$ , then  $[A, B]$  is also in  $\mathcal{E}$ .

The composite

$$L[A, B] \times A \xrightarrow{\varphi^{-1}} L([A, B] \times A) \xrightarrow{L\text{ev}} LB \xrightarrow{\epsilon} B$$

induces a morphism  $c: L[A, B] \rightarrow [A, B]$ . If we can show that  $c\ell: [A, B] \rightarrow [A, B]$  is the identity, then  $c$  will make  $[A, B]$  into an  $L$ -algebra and so  $[A, B]$  will lie in  $\mathcal{E}$ .

Now commutativity of

$$\begin{array}{ccccc} [A, B] \times A & \xrightarrow{\ell \times 1} & L[A, B] \times A & & \\ & \searrow \text{ev} & \searrow \ell & \searrow \varphi^{-1} & \\ & & B & & L([A, B] \times A) \\ & & & \searrow \ell & \downarrow L\text{ev} \\ & & & & LB \\ & & & \searrow 1 & \downarrow \epsilon \\ & & & & B \end{array}$$

shows that  $\text{ev}(c\ell \times 1) = \text{ev}$  and so that  $c\ell = 1$ .  $\square$

**Lemma 4.3.** *If  $L: \mathcal{S} \rightarrow \mathcal{E}$  is semi-left-exact, then  $\mathcal{E}$  is locally cartesian closed if  $\mathcal{S}$  is so.*

*Proof.* For any object  $A \in \mathcal{E}$ , the reflection  $L$  induces a reflection of  $\mathcal{S}/A$  into  $\mathcal{E}/A$ , which is Frobenius. It follows by Proposition 4.2 that  $\mathcal{E}/A$  is cartesian closed.  $\square$

**Remark 4.4.** As observed in the previous section, there are converses to the previous two results. If  $\mathcal{S}$  is cartesian closed, and  $\mathcal{E}$  is a full reflective subcategory closed under exponentials, then the reflection is Frobenius. And if  $\mathcal{S}$  is locally cartesian closed, and  $\mathcal{E}$  is a full reflective subcategory closed under exponentials in the slice categories, then the reflection is semi-left-exact.

We now turn to the existence of weak subobject classifiers. For this, we consider one further condition on our reflection  $L$ , weaker than preservation of finite limits. We say, following [5], that  $L$  is *quasi-lex* if, for each finite diagram  $X: \mathcal{D} \rightarrow \mathcal{S}$ , the canonical comparison map  $L(\lim X) \rightarrow \lim(LX)$  in  $\mathcal{E}$  is both a monomorphism and an epimorphism. We may then say that  $L$  “quasi-preserved” the limit.

The proof of the next result closely follows that of [5, Theorem 1.3.4], although the assumptions made here are rather weaker. When we speak of unions of regular subobjects, we mean unions of subobjects which

happen to be regular: there is no suggestion that the union itself must be regular. We say that such a union is *effective* when it is constructed as the pushout over the intersection.

**Proposition 4.5.** *Let  $\mathcal{S}$  be finitely complete and regular, and suppose further that  $\mathcal{S}$  has (epi, regular mono) factorizations of monomorphisms, and effective unions of regular subobjects; for example,  $\mathcal{S}$  could be a quasitopos. If the reflection  $L: \mathcal{S} \rightarrow \mathcal{E}$  preserves finite products and monomorphisms then it is quasi-lex.*

*Proof.* We know that  $L$  preserves finite products, thus it will suffice to show that it quasi-preserves equalizers.

Consider an equalizer diagram

$$X \xrightarrow{e} Y \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Z$$

in  $\mathcal{S}$ . Since  $L$  preserves finite products, quasi-preservation of this equalizer is equivalent to quasi-preservation of the equalizer

$$X \xrightarrow{e} Y \begin{array}{c} \xrightarrow{\left(\begin{smallmatrix} f \\ Y \end{smallmatrix}\right)} \\ \xrightarrow{\left(\begin{smallmatrix} g \\ Y \end{smallmatrix}\right)} \end{array} Z \times Y$$

in which now the parallel pair has a common retraction, given by the projection  $Z \times Y \rightarrow Y$ . This implies that the exterior of the diagram

$$\begin{array}{ccccc} X & \xrightarrow{e} & Y & & \\ \downarrow e & & \downarrow f' & \searrow & \\ Y & \xrightarrow{g'} & Z' & \xrightarrow{m} & Z \times Y \\ & \searrow & \downarrow & \nearrow & \\ & & \left(\begin{smallmatrix} g \\ Y \end{smallmatrix}\right) & & \left(\begin{smallmatrix} f \\ Y \end{smallmatrix}\right) \end{array}$$

is a pullback. By effectiveness of unions, we can form the union of  $\left(\begin{smallmatrix} f \\ Y \end{smallmatrix}\right)$  and  $\left(\begin{smallmatrix} g \\ Y \end{smallmatrix}\right)$  by constructing the pushout square as in the interior of the diagram. Then the induced map  $m: Z' \rightarrow Z \times Y$  will be the union, and in particular is a monomorphism. Now apply the reflection  $L$  to

this last diagram, to get a diagram

$$\begin{array}{ccccc}
 LX & \xrightarrow{Le} & LY & & \\
 Le \downarrow & & Lf' \downarrow & \searrow & \\
 LY & \xrightarrow{Lg'} & LZ' & \xrightarrow{Lm} & LZ \times LY \\
 & \searrow & & \nearrow & \\
 & & & & (Lg)_{LY}
 \end{array}$$

$(Lf)_{LY}$

in  $\mathcal{S}$ . The interior square is still a pushout, and  $Lm$  and  $Le$  are still monomorphisms. Factorize  $Le$  as an epimorphism  $k: LX \rightarrow A$  followed by a regular monomorphism  $d: A \rightarrow LY$ . Then  $d$ , like any regular monomorphism, is the equalizer of its cokernel pair. Since  $k$  is an epimorphism,  $d$  and  $dk = Le$  have the same cokernel pair, namely  $Lf'$  and  $Lg'$ . Thus  $d$  is the equalizer of  $Lf'$  and  $Lg'$ , and  $k$  is the canonical comparison. It is an epimorphism by construction, and a monomorphism by the standard cancellation properties. Thus  $L$  quasi-preserves the equalizer of  $f'$  and  $g'$ , and so also the equalizer of  $\begin{pmatrix} f \\ g \end{pmatrix}$  and  $\begin{pmatrix} g \\ f \end{pmatrix}$ , and so finally that of  $f$  and  $g$ .  $\square$

**Remark 4.6.** In fact there is also a partial converse to the preceding result: if  $L$  is quasi-lex and has strongly epimorphic units, then it preserves finite products and monomorphisms; indeed any quasi-lex  $L$  preserves monomorphisms: see [5].

**Lemma 4.7.** *If  $L$  is quasi-lex, then  $\mathcal{E}$  has a weak subobject classifier if  $\mathcal{S}$  does so.*

*Proof.* Let  $t: 1 \rightarrow \Omega$  be the weak subobject classifier of  $\mathcal{S}$ . Now  $Lt: L1 \rightarrow L\Omega$  is a strong (in fact split) subobject, so there is a unique map  $\chi: L\Omega \rightarrow \Omega$  such that

$$\begin{array}{ccc}
 L1 & \xrightarrow{Lt} & L\Omega \\
 \downarrow & & \downarrow \chi \\
 1 & \xrightarrow{t} & \Omega
 \end{array}$$

is a pullback. Form the equalizer

$$\Omega' \xrightarrow{i} \Omega \xrightleftharpoons[1]{\chi^\ell} \Omega$$

in  $\mathcal{S}$ .



Observe that  $\chi.\ell.\chi = \chi.L\chi.\ell L\Omega = \chi.L\chi.L\ell\Omega$ , and so  $\chi.\ell.\chi.Li = \chi.L\chi.L\ell\Omega.Li = \chi.Li$ ; thus  $\chi.Li$  factorizes as  $i.\chi'$  for a unique  $\chi': L\Omega' \rightarrow \Omega'$ .

Furthermore,  $i.\chi'.\ell\Omega' = \chi.Li.\ell\Omega' = \chi.\ell.i = i$  and so  $\chi'.\ell = 1$ . This proves that  $\Omega' \in \mathcal{E}$ . Furthermore  $\chi.\ell.t = \chi.Lt.\ell = t$  and so  $t = it'$  for a unique  $t': 1 \rightarrow \Omega'$ . We shall show that  $t': 1 \rightarrow \Omega'$  is a weak subobject classifier for  $\mathcal{E}$ .

Suppose then that  $m: A \rightarrow B$  is a strong subobject in  $\mathcal{E}$ . The inclusion, being a right adjoint, preserves strong subobjects, so there is a unique  $f: B \rightarrow \Omega$  for which the diagram

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ \downarrow & & \downarrow f \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

is a pullback. We shall show that  $f$  factorizes as  $f = if'$ ; it then follows that  $f': B \rightarrow \Omega'$  is the unique map in  $\mathcal{E}$  classifying  $m$ .

To do so, it will suffice to show that  $\chi.\ell.f = f$ , or equivalently  $\chi.Lf.\ell = f$ . Now consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ \ell \downarrow & & \downarrow \ell \\ LA & \xrightarrow{Lm} & LB \\ L! \downarrow & & \downarrow Lf \\ L1 & \xrightarrow{Lt} & L\Omega \\ \downarrow & & \downarrow \chi \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

in which the top square is a pullback since  $\ell A$  and  $\ell B$  are invertible, and the bottom square is a pullback, by definition of  $\chi$ . Thus if the middle square is a pullback, then the composite will be, and so  $\chi.Lf.\ell$  must be the unique map  $f$  classifying  $m$ .

Now we know that the comparison  $x$  from  $LA$  to the pullback  $P$  of  $Lf$  and  $Lt$  is both an epimorphism and a monomorphism in  $\mathcal{E}$ . But  $Lm$ , like  $m$ , is a strong monomorphism, and factorizes as  $sx$ , where  $s$  is the pullback of  $Lt$ ; thus  $x$  is a strong monomorphism and an epimorphism, and so invertible. This completes the proof.  $\square$

Combining the main results of this section, we have:

**Theorem 4.8.** *If the reflection  $L: \mathcal{S} \rightarrow \mathcal{E}$  is semi-left-exact and quasi-lex, then  $\mathcal{E}$  is a quasitopos if  $\mathcal{S}$  is one.*

**Corollary 4.9.** *If the reflection  $L: \mathcal{S} \rightarrow \mathcal{E}$  is semi-left-exact and preserves finite products and monomorphisms, and so also if it has stable units and preserves monomorphisms, then  $\mathcal{E}$  is a quasitopos if  $\mathcal{S}$  is one.*

*Proof.* Combine the previous theorem with Theorem 3.5 and Proposition 4.5.  $\square$

## 5. SUBQUASITOPPOSES

As recalled in the introduction, a *subtopos* of a topos is a full reflective subcategory for which the reflection preserves finite limits. These can be characterized in various ways, for example using Lawvere-Tierney topologies, or universal closure operators. By analogy with this, we define a *subquasitopos* of a quasitopos  $\mathcal{S}$  to be a full reflective subcategory for which the reflection has stable units and preserves monomorphisms. By Corollary 4.9 we know that the subcategory will indeed be a quasitopos. In this section, we give a classification of subquasitoposes of  $\mathcal{S}$  using *proper universal closure operators*.

A closure operator  $j$ , on a category  $\mathcal{C}$  with finite limits, assigns to each subobject  $A' \leq A$  a subobject  $j(A') \leq A$  in such a way that  $A' \leq j(A') = j(j(A'))$  and if  $A_1 \leq A_2 \leq A$  then  $j(A_1) \leq j(A_2) \leq A$ . The closure operator is said to be *universal* if for each  $f: B \rightarrow A$  and each  $A' \leq A$  we have  $f^*(j(A')) = j(f^*(A'))$ . It is said to be *proper*, especially in the case where  $\mathcal{C}$  is a quasitopos, if  $j(A') \leq A$  is strong subobject whenever  $A' \leq A$  is one; of course this is automatic if  $\mathcal{C}$  is a topos, so that all subobjects are strong. If  $j(A') \leq A$  is a strong subobject for *all* subobjects  $A' \leq A$ , then  $j$  is said to be a *strict universal closure operator*.

Given a universal closure operator  $j$  on  $\mathcal{C}$ , a subobject  $m: A' \rightarrow A$  is said to be  *$j$ -dense* if  $j(A' \leq A) = A$ . An object  $X$  of  $\mathcal{C}$  is said to be a  *$j$ -sheaf* if it is orthogonal to each  $j$ -dense monomorphism, and  *$j$ -separated* if it is separated with respect to each  $j$ -dense morphism.

Recall, for example from [12, Theorem A4.4.8], that for a quasitopos  $\mathcal{S}$  there is a bijection between localizations of  $\mathcal{S}$  and proper universal closure operators on  $\mathcal{S}$ . Explicitly, the bijection associates to a proper universal closure operator  $j$  the subcategory  $\text{Sh}(\mathcal{S}, j)$  of  $j$ -sheaves; while for a localization  $L: \mathcal{S} \rightarrow \mathcal{E}$ , the corresponding closure operator sends a subobject  $A' \leq A$  to the pullback of  $LA' \leq LA$  along the unit  $\ell: A \rightarrow LA$ . Furthermore, by [12, Theorem A4.4.5], if  $j$  is strict then  $\text{Sh}(\mathcal{S}, j)$  is a topos. Conversely, if  $j$  is a proper universal closure

operator for which  $\mathrm{Sh}(\mathcal{S}, j)$  is a topos, then for any subobject  $A' \leq A$  in  $\mathcal{S}$ , the reflection  $LA' \leq LA$  is a subobject in a topos, hence a strong subobject; thus  $j(A') \leq A$  too is a strong subobject, and  $j$  is strict.

For any quasitopos  $\mathcal{Q}$ , there is a strict universal closure operator sending a subobject  $A' \leq A$  to its strong closure  $\overline{A'} \leq A$ , given by factorizing the inclusion  $A' \rightarrow A$  as an epimorphism  $A' \rightarrow \overline{A'}$  followed by a strong monomorphism  $\overline{A'} \rightarrow A$ . An object of  $\mathcal{Q}$  is said to be *coarse* if it is a sheaf for this closure operator, and we write  $\mathrm{Cs}(\mathcal{Q})$  for the full subcategory consisting of the coarse objects; this is a topos, and is reflective in  $\mathcal{Q}$  via a finite-limit-preserving reflection  $\mathcal{Q} \rightarrow \mathrm{Cs}(\mathcal{Q})$  which inverts precisely those monomorphisms which are also epimorphisms; see [12, A4.4].

We now suppose that  $\mathcal{S}$  is a quasitopos, and  $L: \mathcal{S} \rightarrow \mathcal{E}$  a reflection onto a subquasitopos. As before, we write  $\mathcal{R}$  for the full subcategory of  $\mathcal{S}$  consisting of those objects  $X \in \mathcal{S}$  for which the unit  $\ell: X \rightarrow LX$  is a monomorphism. We saw in Proposition 3.3 that  $\mathcal{R}$  is reflective in  $\mathcal{S}$ , and we saw in Corollary 3.4 that this reflection has stable units and preserves monomorphisms; thus by Corollary 4.9 the category  $\mathcal{R}$ , like  $\mathcal{E}$ , is a quasitopos.

**Proposition 5.1.** *There is a strict universal closure operator  $k$  on  $\mathcal{S}$  whose sheaves are the coarse objects in  $\mathcal{E}$  and whose separated objects are the objects of  $\mathcal{R}$ . The class  $\mathcal{K}$  of  $k$ -dense monomorphisms consists of all those monomorphisms  $m: X \rightarrow Y$  for which the monomorphism  $Lm: LX \rightarrow LY$  is also an epimorphism in  $\mathcal{E}$ .*

*Proof.* Write  $H: \mathcal{E} \rightarrow \mathrm{Cs}(\mathcal{E})$  for the reflection; by the remarks above it preserves finite limits. Recall from Proposition 4.5 that  $L: \mathcal{S} \rightarrow \mathcal{E}$  is quasi-lex; since  $H$  preserves finite limits and inverts the epimorphic monomorphisms, the composite  $HL: \mathcal{S} \rightarrow \mathrm{Cs}(\mathcal{E})$  is a finite-limit-preserving reflection. It follows that there is a proper universal closure operator  $k$  on  $\mathcal{S}$  whose sheaves are the coarse objects in  $\mathcal{E}$ . Since  $\mathrm{Cs}(\mathcal{E})$  is a topos,  $k$  is strict.

A monomorphism  $m: X \rightarrow Y$  in  $\mathcal{S}$  is  $k$ -dense just when it is inverted by  $HL$ ; that is, just when the monomorphism  $Lm$  is also an epimorphism. An object  $A \in \mathcal{E}$  is certainly separated with respect to such an  $m$ , since for any  $a: X \rightarrow A$  the induced  $La: LX \rightarrow A$  has at most one factorization through the epimorphism  $Lm: LX \rightarrow LY$ . Furthermore, the  $m$ -separated objects are closed under subobjects, so that every object of  $\mathcal{R}$  is  $k$ -separated.

Conversely, suppose that  $X$  is  $k$ -separated; that is, separated with respect to each  $k$ -dense  $m$ . We must show that  $\ell X: X \rightarrow LX$  is a monomorphism. Let  $d, c: K \rightrightarrows X$  be the kernel pair of  $\ell X$ , and

$\delta: X \rightarrow K$  the diagonal. If  $X$  is  $\delta$ -separated, then since  $d\delta = 1 = c\delta$ , the two morphisms  $d$  and  $c$  must be equal, which is to say that  $\ell X$  is a monomorphism. Thus it will suffice to show that  $\delta$  is  $k$ -dense. Since  $L$  preserves finite products and monomorphisms, it also preserves jointly monomorphic pairs; thus  $Ld$  and  $Lc$  are, like  $d$  and  $c$ , jointly monomorphic. On the other hand  $L\ell X$  is invertible, and  $L\ell X.Ld = L\ell X.Lc$ , and so  $Ld = Lc$ ; thus in fact  $Ld$  is monomorphic. But  $L\delta$  is a section of  $Ld$ , and so both maps are invertible. In particular, since  $L\delta$  is invertible,  $\delta$  is  $k$ -dense, and so  $X \in \mathcal{R}$ .  $\square$

We are now ready to prove our characterization of subquasitoposes.

**Theorem 5.2.** *Subquasitoposes of a quasitopos  $\mathcal{S}$  are in bijection with pairs  $(h, k)$ , where  $k$  is a strict universal closure operator on  $\mathcal{S}$ , and  $h$  is a proper universal closure operator on  $\text{Sep}(\mathcal{S}, k)$  with the property that every  $h$ -dense subobject is also  $k$ -dense; the subquasitopos corresponding to the pair  $(h, k)$  is  $\text{Sh}(\text{Sep}(\mathcal{S}, k), h)$ .*

*Proof.* If  $k$  is a strict universal closure operator on  $\mathcal{S}$ , then the category  $\text{Sh}(\mathcal{S}, k)$  of  $k$ -sheaves is reflective in  $\mathcal{S}$  via a finite-limit-preserving reflection  $M$ . The category  $\text{Sep}(\mathcal{S}, k)$  of  $k$ -separated objects is also reflective, and we may obtain the reflection  $M'$  by factorizing the unit  $m: X \rightarrow MX$  of  $M$  as a strong epimorphism  $m': X \rightarrow M'X$  followed by a monomorphism  $\kappa: M'X \rightarrow MX$ , exactly as in Proposition 3.3. By Corollary 3.4 we know that  $M'$  has stable units and preserves monomorphisms. Now  $\text{Sh}(\text{Sep}(\mathcal{S}, k), h)$  is reflective in  $\text{Sep}(\mathcal{S}, k)$  via a finite-limit-preserving reflection, and so the composite reflection  $\mathcal{S} \rightarrow \text{Sh}(\text{Sep}(\mathcal{S}, k), h)$  has stable units and preserves monomorphisms.

Conversely, let  $L: \mathcal{S} \rightarrow \mathcal{E}$  be a reflection onto a subquasitopos. As above, we define  $k$  to be the strict universal closure operator whose sheaves are the coarse objects in  $\mathcal{E}$ . By Corollary 3.4, we know that the restriction  $\bar{L}: \mathcal{R} \rightarrow \mathcal{E}$  of  $L$  to  $\mathcal{R}$  preserves finite limits, and so corresponds to a proper universal closure operator  $h$  on  $\mathcal{R}$ , whose category of sheaves is  $\mathcal{E}$ . Since every  $k$ -sheaf is an  $h$ -sheaf, every  $h$ -dense monomorphism is  $k$ -dense.

It remains to prove the uniqueness of the  $h$  and  $k$  giving rise to  $L: \mathcal{S} \rightarrow \mathcal{E}$  as in the first paragraph. We constructed  $M'$  above by factorizing  $X \rightarrow MX$  as a strong epimorphism  $m': X \rightarrow M'X$  followed by a monomorphism  $\kappa: M'X \rightarrow MX$ . Since every  $h$ -dense monomorphism is  $k$ -dense, certainly every  $k$ -sheaf is an  $h$ -sheaf. Thus  $\kappa: M'X \rightarrow MX$  factorizes through  $LX$  by some  $\nu: M'X \rightarrow LX$ , necessarily monic, and now  $\ell: X \rightarrow LX$  factorizes as a strong epimorphism  $m': X \rightarrow M'X$  followed by a monomorphism  $\nu: M'X \rightarrow LX$ . Thus

$\text{Sep}(\mathcal{S}, k)$  is uniquely determined by  $L$ . In general, there can be several different proper universal closure operators with a given category of separated objects, but by the discussion after [12, Theorem A4.4.8], there can be at most one *strict* universal closure operator with a given category of separated objects. Thus  $k$  is uniquely determined. Unlike the case of separated objects, a proper universal closure operator *is* uniquely determined by its sheaves, and so  $h$  is also uniquely determined.  $\square$

Observe that in our characterization the two proper universal closure operators live on different categories. In the next section, we shall see that when  $\mathcal{S}$  is a presheaf topos, there is an alternative characterization in terms of two universal closure operators on  $\mathcal{S}$ . In fact, even for a general quasitopos, we may give a characterization purely in terms of structure existing in  $\mathcal{S}$  provided that we prepared to work with stable classes of monomorphisms rather than universal closure operators.

As in Proposition 5.1, we let  $\mathcal{K}$  denote the class of monomorphisms  $m: X \rightarrow Y$  in  $\mathcal{S}$  for which  $Lm$  is an epimorphism as well as a monomorphism; as there, these are the dense monomorphisms for a universal closure operator, and so in particular are stable under pullback. Now we let  $\mathcal{J}$  be the class of monomorphisms  $m: X \rightarrow Y$  in  $\mathcal{S}$ , every pullback of which is inverted by  $L$ . This is clearly the largest stable class of monomorphisms inverted by  $L$ . As we saw in Proposition 3.3, the unit  $\ell X: X \rightarrow LX$  is a monomorphism for any  $X \in \mathcal{R}$ ; furthermore since  $L$  has stable units, it preserves the pullback of  $\ell X$  along any map, and so  $L$  inverts not just  $\ell X$  but also all of its pullbacks. Thus  $\ell X$  lies in  $\mathcal{J}$  for all  $X \in \mathcal{R}$ ; more generally, since  $L$  preserves all pullbacks over objects in  $\mathcal{R}$  by Remark 3.6, any monomorphism  $f: X \rightarrow Y$  in  $\mathcal{R}$  which is inverted by  $L$  will lie in  $\mathcal{J}$ .

**Theorem 5.3.** *Let  $\mathcal{S}$  be a quasitopos, and  $L: \mathcal{S} \rightarrow \mathcal{E}$  a reflection onto a full subcategory. If  $L$  has stable units and preserves monomorphisms, then*

- (i) *an object  $X$  of  $\mathcal{S}$  lies in  $\mathcal{R}$  just when it is  $\mathcal{K}$ -separated;*
- (ii) *an object  $X$  of  $\mathcal{S}$  lies in  $\mathcal{E}$  just when it is  $\mathcal{K}$ -separated and a  $\mathcal{J}$ -sheaf.*

*Proof.* We have already proved part (i) in Proposition 5.1. For part (ii), first observe that if  $A \in \mathcal{E}$  then  $A$  is orthogonal to all morphisms inverted by  $L$ , not just those in  $\mathcal{J}$ . It is of course also separated with respect to  $\mathcal{K}$ .

Conversely, if  $A$  is  $\mathcal{K}$ -separated then it is in  $\mathcal{R}$ ; but then  $\ell A: A \rightarrow LA$  is in  $\mathcal{J}$ , and so if  $A$  is a  $\mathcal{J}$ -sheaf then  $\ell A$  must be invertible and so  $A \in \mathcal{E}$ .  $\square$

At the current level of generality, there seems no reason why  $\mathcal{J}$  need be the dense monomorphisms for a proper universal closure operator on  $\mathcal{S}$ . In the following section we shall see that this will be so if  $\mathcal{S}$  is a presheaf topos.

## 6. GROTHENDIECK QUASITOPPOSES

In this final section we suppose that  $\mathcal{S}$  is a presheaf topos  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ , as well as the standing assumption that  $L: \mathcal{S} \rightarrow \mathcal{E}$  is a reflection which preserves monomorphisms and has stable units. Recall that  $\mathcal{J}$  consists of the monomorphisms which are stably inverted by the reflection  $L$ , and that  $\mathcal{K}$  consists of the monomorphisms  $m$  for which  $Lm$  is an epimorphism in  $\mathcal{E}$  as well as a monomorphism. By Theorem 5.2, the class  $\mathcal{K}$  consists of the dense monomorphisms for a (proper) universal closure operator  $k$  on  $\mathcal{S}$ ; and by our new assumption that  $\mathcal{S}$  is a presheaf topos,  $k$  corresponds to a Grothendieck topology with the same sheaves and separated objects. Since at this stage we are really only interested in the sheaves and separated objects, we take the liberty of using the same name  $k$  for the topology as for the universal closure operator.

As for  $\mathcal{J}$ , since it is a stable system of monomorphisms, it can be seen as a coverage, in the sense of [12], and so generates a Grothendieck topology  $j$  whose sheaves are the objects orthogonal to  $\mathcal{J}$ .

**Theorem 6.1.** *For a reflection  $L: [\mathcal{C}^{\text{op}}, \mathbf{Set}] \rightarrow \mathcal{E}$  onto a full subcategory of a presheaf category, the following conditions are equivalent:*

- (i) *The subcategory  $\mathcal{E}$  has the form  $\text{Sep}(k) \cap \text{Sh}(j)$  for topologies  $j$  and  $k$  on  $\mathcal{C}$  with  $k$  containing  $j$ ;*
- (ii)  *$L$  is semi-left-exact and preserves finite products and monomorphisms;*
- (iii)  *$L$  has stable units and preserves monomorphisms.*

An  $\mathcal{E}$  as in the theorem is called a Grothendieck quasitopos; as we saw in the introduction, a category  $\mathcal{E}$  has this form for some  $\mathcal{C}$ ,  $j$ , and  $k$  if and only if it is a locally presentable quasitopos [2].

*Proof.* The equivalence of (ii) and (iii) was shown in Theorem 3.5. The fact that these imply (i) now follows from Theorem 5.3. Thus it will suffice to suppose (i) and show that (iii) follows.

We have adjunctions

$$\mathrm{Sep}(k) \cap \mathrm{Sh}(j) \begin{array}{c} \xleftarrow{L_2} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathrm{Sh}(j) \begin{array}{c} \xleftarrow{L_1} \\ \perp \\ \xrightarrow{\quad} \end{array} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$

and  $L_1$  preserves all finite limits. It will clearly suffice to show that  $L_2$  preserves monomorphisms as well as pullbacks over an object of  $\mathrm{Sep}(k) \cap \mathrm{Sh}(j)$ .

Now  $\mathrm{Sep}(k) \cap \mathrm{Sh}(j)$  is just the category of separated objects in the topos  $\mathrm{Sh}(j)$  for a (Lawvere-Tierney) topology  $k'$  in  $\mathrm{Sh}(j)$ . Thus it will suffice to show that for a topos  $\mathcal{S}$  and a topology  $k$ , the reflection  $L: \mathcal{S} \rightarrow \mathrm{Sep}(k)$  preserves monomorphisms as well as pullbacks over separated objects. This is the special case of (one direction of) Theorem 5.2, where  $h$  is trivial.  $\square$

As we saw in the previous section, the topology  $k$  can be recovered from  $\mathrm{Sep}(k) \cap \mathrm{Sh}(j)$ , since  $\mathrm{Sh}(k)$  is the topos of coarse objects in  $\mathrm{Sep}(k) \cap \mathrm{Sh}(j)$ , which can be obtained by inverting all those morphisms in  $\mathrm{Sep}(k) \cap \mathrm{Sh}(j)$  which are both monomorphisms and epimorphisms. Unlike the case of the (proper) universal closure operator  $h$  of the previous section,  $j$  need not be uniquely determined, as we now explain.

There exist non-trivial topologies  $k$  for which every separated object is a sheaf; these were studied by Johnstone in [11]. In this case, for *any* topology  $j$  contained in  $k$  we have

$$\mathrm{Sep}(k) \cap \mathrm{Sh}(j) = \mathrm{Sh}(k) \cap \mathrm{Sh}(j) = \mathrm{Sh}(k),$$

where the last step holds since  $\mathrm{Sh}(k) \subseteq \mathrm{Sh}(j)$ . In particular we could take  $j$  to be either trivial or  $k$  and obtain the same subcategory  $\mathrm{Sh}(k)$  as  $\mathrm{Sep}(k) \cap \mathrm{Sh}(j)$ .

**Example 6.2.** For example, as explained in [12, Example A4.4.9], we could take the category  $\mathbf{Set}^M$  of  $M$ -sets, where  $M$  is the two-element monoid  $M = \{1, e\}$ , with  $e^2 = e$ , or equivalently the category of sets equipped with an idempotent. Then  $\mathbf{Set}$  can be seen as the full reflective subcategory of  $M$ -sets with trivial action. The reflection  $L: \mathbf{Set}^M \rightarrow \mathbf{Set}$  splits the idempotent; this preserves all limits and so is certainly a localization. Since the unit of the adjunction is epimorphic, every separated object for the induced topology  $k$  is a sheaf.

**Remark 6.3.** Theorem 6.1 can be generalized to the case of a Grothendieck topos  $\mathcal{S}$  in place of  $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ ; then  $j$  and  $k$  would be Lawvere-Tierney topologies on  $\mathcal{S}$ . It can further be generalized to the case where  $\mathcal{S}$  is a Grothendieck quasitopos, provided that we are willing to work with proper universal closure operators  $j$  and  $k$  rather than



topologies. In either case,  $\mathcal{E}$  will still be a quasitopos by Corollary 4.9, and is in fact a Grothendieck quasitopos. In the case of a quasitopos or topos  $\mathcal{S}$  which is not locally presentable, however, there seems no reason why the objects orthogonal to  $\mathcal{J}$  should be the sheaves, either for a topology or a universal closure operator.

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